ON FREE ASSOCIATIVE ALGEBRAS LINEARLY GRADED BY FINITE GROUPS

VITOR O. FERREIRA AND LUCIA S. I. MURAKAMI

Abstract. As an instance of a linear action of a Hopf algebra on a free associative algebra, we consider finite group gradings of a free algebra induced by gradings on the space spanned by the free generators. The homogeneous component corresponding to the identity of the group is a free subalgebra which is graded by the usual degree. We look into its Hilbert series and prove that it is a rational function by giving an explicit formula. As an application, we show that, under suitable conditions, this subalgebra is finitely generated if and only if the grading on the base vector space is trivial.

Introduction

Let k be a field and let V be a vector space over k. We shall denote by $T(V)$ the *tensor algebra* on V. It is defined by

$$T(V) = \bigoplus_{n\geq 0} V^{\otimes n},$$

where $V^{\otimes 0} = k$ and $V^{\otimes n} = V^{\otimes n-1} \otimes_k V$ for all $n \geq 1$, with multiplication satisfying

$$(v_1 \otimes \cdots \otimes v_n)(u_1 \otimes \cdots \otimes u_m) = v_1 \otimes \cdots \otimes v_n \otimes u_1 \otimes \cdots \otimes u_m,$$

for all $n, m \geq 1$ and $v_i, u_j \in V$. If X is a basis of V over k, then there is a natural isomorphism between $T(V)$ and the free associative algebra $k\langle X\rangle$ on X over k. So if $\dim V = d$, then $T(V)$ is a free algebra of rank d over k.

Now let H be a Hopf algebra over k and suppose that V is a left H-module. The action of H on V induces a structure of an H-module algebra on $T(V)$ such that

$$h \cdot (v_1 \otimes \cdots \otimes v_n) = \sum_{(h)} (h_{(1)} \cdot v_1) \otimes \cdots \otimes (h_{(n)} \cdot v_n),$$

for all $h \in H$, $n \geq 1$ and $v_i \in V$. Such Hopf algebra actions on $T(V)$ will be called *linear actions*. We shall say that a linear action is *scalar* whenever it is induced by a scalar action of H on V, that is to say, whenever for every $h \in H$, there exists a scalar $\lambda = \lambda(h) \in k$ such that $h \cdot v = \lambda v$, for all $v \in V$.

Given a Hopf algebra H and an arbitrary H-module algebra A, the subset

$$A^H = \{a \in A : h \cdot a = \varepsilon(h)a, \text{ for all } h \in H\}$$

Date: 8 November 2008.

2000 *Mathematics Subject Classification.* 16S10, 16W30, 16W50.

Key words and phrases. Free associative algebra; Hopf algebra actions; group gradings; invariants.

The first author was partially supported by CNPq (Grant 308163/2007-9) and by PRP-USP (ProIP Proc. 2006.1.24191.1.6).

The second author was partially supported by PRP-USP (ProIP Proc. 2006.1.24191.1.6).

is a subalgebra, called the *subalgebra of invariants* of A under the action of H.

The study of the subalgebra of invariants $T(V)^H$ under the action of a Hopf algebra H has received attention recently. It is a natural extension of the so-called noncommutative invariant theory, which concerns invariants of a free algebra under the action of a group of linear automorphisms. Lane [La76] and Kharchenko [Kh78] showed that if G is a group of linear automorphisms of a free algebra $T(V)$ then the subalgebra of invariants $T(V)^G$ of $T(V)$ under the action of G is again free. This has been generalized in [FMP04], where it is shown that the subalgebra of invariants $T(V)^H$ under a linear action of an arbitrary Hopf algebra is free. In fact, for a finite-dimensional pointed Hopf algebra H acting linearly on a free algebra $T(V)$, in [FMP04] a Galois correspondence between the subalgebras of H which are right coideals and the free subalgebras of $T(V)$ containing $T(V)^H$ is built. This correspondence generalizes Kharchenko's correspondence in [Kh78] for groups of linear automorphisms.

Regarding the rank of $T(V)$, in the case of a finite group G of linear automorphisms of $T(V)$, Dicks and Formanek [DF82] and Kharchenko [Kh84] have shown that $T(V)^G$ is a finitely generated algebra if and only if each element of G is a scalar automorphism. In [FM07], the authors use the Galois correspondence of [FMP04] to show that the same happens in a more general setting. Precisely, if H is a finite-dimensional Hopf algebra generated by grouplike and skew-primitive elements which act linearly on a free algebra $T(V)$ of finite rank, then $T(V)^H$ is finitely generated as an algebra if and only if the action of H is in fact scalar.

The free algebra $T(V)$ has a natural ($\mathbb{N}$-)grading, induced by the usual degree function on $T(V)$, under which the homogeneous component of degree n, for a positive integer n, is the span of the monomials of length n on the elements of V. If H is a Hopf algebra acting linearly on $T(V)$, then this grading is inherited by $T(V)^H$, for the action of H is clearly homogeneous.

One can attach to a general graded algebra $A = \bigoplus_{n\geq 0} A_n$ such that A_n is a finite-dimensional vector subspace a formal power series over $\mathbb{Z}$, its *Hilbert (or Poincaré) series* $P(A,t)$, defined by

$$P(A,t) = \sum_{n\geq 0} (\dim A_n) t^n.$$

This is a combinatorial object that codifies quantitative information on the homogeneous components of A. It is an elementary fact that given a vector space V of finite dimension d, the Hilbert series of its tensor algebra is a very simple rational function given by

$$P(T(V),t) = \frac{1}{1-dt}.$$

Thus, given a Hopf algebra acting linearly on a free algebra $T(V)$ of finite rank, it is of interest to investigate the nature of the Hilbert series of the subalgebra of invariants $T(V)^H$.

In [DF82], Dicks and Formanek give explicit formulas (depending on the characteristic of k) for the Hilbert series of the subalgebra of invariants $T(V)^G$ of a free algebra of finite rank $T(V)$ under the action of a finite group G of linear automorphisms.

In this paper we restrict to actions of dual Hopf algebras of group algebras on free algebras, that is to say, we consider free algebras graded by finite groups. So if

$T(V)$ is a linearly G-graded algebra for a finite group G, in Section 1 we describe the grading by the degree function that $T(V)_e$ inherits from this grading on $T(V)$, where e stands for the identity element of G. Section 2 is devoted to the deduction of an explicit formula for the Hilbert series of $T(V)_e$. Finally, in Section 3, we apply the results of the previous section in order to obtain a criterium for finite generation of $T(V)_e$ in terms of the G-grading.

We shall make use of the usual definitions and notation of Hopf algebra theory as found in [Mo93] or [DNR01].

1. Linear group gradings on free algebras

Let G be a finite group and let kG be the Hopf group algebra of G over k. Now let H be the dual Hopf algebra $(kG)^*$ of kG. So, H is the vector space of all linear functionals on kG with multiplication given by

$$\langle \alpha\beta, x\rangle = \langle \alpha, x\rangle\langle \beta, x\rangle, \quad \text{for all } \alpha, \beta \in H \text{ and } x \in G,$$

and comultiplication satisfying

$$\Delta(p_x) = \sum_{yz=x} p_y \otimes p_z, \quad \text{for all } x \in G,$$

where $\{p_x : x \in G\}$ is the dual basis of the basis G of kG, *i.e.*, one has

$$p_x(y) = \delta_{x,y}, \quad \text{for all } x, y \in G.$$

The counit of H is just the augmentation map $\varepsilon \colon kG \to k$.

For this Hopf algebra, it is a well-known fact that an algebra A is an H-module algebra if and only if A is *G-graded*, that is to say, there exists a family $\{A_x : x \in G\}$ of subspaces of A satisfying

(1) $1_A \in A_e$, where 1_A stands for the unity of A and e for the identity element of G, and
(2) $A_x A_y \subseteq A_{xy}$, for all $x \in G$.

Moreover, when this is the case, $A^H = A_e$, the homogeneous component of the grading of A associated to the identity element of e, henceforth referred to as the *identity component*.

Now let V be a vector space of finite dimension d over k and suppose that $H = (kG)^*$ acts linearly on $T(V)$. So we have an action of H on V which induces the action on $T(V)$. This amounts to saying that we are given a decomposition $V = \bigoplus_{x\in G} V_x$ of V as a direct sum of subspaces indexed by the elements of G — when this is the case we shall say that V is a *G-graded vector space* — and that $T(V)$ has a structure of a G-graded algebra induced by this decomposition. More specifically, we have a decomposition of $T(V)$ given by

$$T(V) = \bigoplus_{x\in G} T(V)_x,$$

where for each $x \in G$, $x \neq e$, the subspace $T(V)_x$ is given by

$$T(V)_x = \bigoplus_{y_1\cdots y_n = x} V_{y_1} \otimes \cdots \otimes V_{y_n}, \quad \text{for all } n \geq 1 \text{ and } y_i \in G,$$

and

$$T(V)_e = k \oplus \bigoplus_{y_1\cdots y_n = e} V_{y_1} \otimes \cdots \otimes V_{y_n}, \quad \text{for all } n \geq 1 \text{ and } y_i \in G.$$

As we have seen above, $T(V)_e$ is a free subalgebra of $T(V)$.

2. The Hilbert series of a linearly graded free algebra

For the decomposition $V = \bigoplus_{x\in G} V_x$, write $d_x = \dim V_x$. We shall look at the Hilbert series $P(T(V)_e, t)$ of the graded subalgebra $T(V)_e$.

We start by establishing a recursive relation among the coefficients of $P(T(V)_e, t)$ and coefficients of the Hilbert series of the remaining homogeneous components of $T(V)$.

For each $n \geq 1$, we have

$$V^{\otimes n} = \bigoplus_{x\in G} \left(\bigoplus_{y_1 \dots y_n = x} V_{y_1} \otimes \cdots \otimes V_{y_n} \right).$$

For each $x \in G$, let

$$(V^{\otimes n})_x = \bigoplus_{y_1 \dots y_n = x} V_{y_1} \otimes \cdots \otimes V_{y_n} \tag{1}$$

and let $a_n^{(x)} = \dim (V^{\otimes n})_x$.

Lemma 1. *The numbers $a_n^{(x)}$ defined above satisfy the following recursive relations*

$$a_n^{(x)} = \sum_{y\in G} d_{xy^{-1}} a_{n-1}^{(y)}, \quad \textit{for } n \geq 1, \tag{2}$$

where $a_0^{(x)} = \delta_{e,x}$.

Proof. From (1), we get, for each $n \geq 2$ and $x \in G$,

$$(V^{\otimes n})_x = \bigoplus_{y\in G} V_{xy^{-1}} \otimes \left(\bigoplus_{z_1 \dots z_{n-1} = y} V_{z_1} \otimes \cdots \otimes V_{z_{n-1}} \right) = \bigoplus_{y\in G} V_{xy^{-1}} \otimes (V^{\otimes n-1})_y.$$

This implies (2) for $n \geq 2$. The other relations are trivial. □

Theorem 2. *Let G be a finite group and let V be a finite-dimensional G-graded vector space. Then the Hilbert series $P(T(V)_e, t)$ of the identity component of the G-grading on $T(V)$ induced by the G-grading on V is a rational function of the form*

$$P(T(V)_e, t) = \frac{p(t)}{(1 - dt)q(t)},$$

where $d = \dim V$ and $p(t)$ and $q(t)$ are polynomials with integer coefficients with $\deg p(t), \deg g(t) \leq |G| - 1$.

Proof. Suppose $V = \bigoplus_{x\in G} V_x$. We shall use the notation preceding Lemma 1. For each $x \in G$, let $F_x(t)$ be the power series in $\mathbb{Z}[[t]]$ defined by $F_x(t) = \sum_{n\geq 0} a_n^{(x)} t^n$. Note that $F_e(t) = P(T(V)_e, t)$. By (2), we have

$$F_x(t) = \sum_{y\in G} \left(\sum_{n\geq 1} d_{xy^{-1}} a_{n-1}^{(y)} t^n \right) \quad \text{if } x \neq e, \text{ and}$$

$$F_e(t) = 1 + \sum_{y\in G} \left(\sum_{n\geq 1} d_{y^{-1}} a_{n-1}^{(y)} t^n \right).$$

Therefore, these series are related by

$$F_x(t) = \sum_{y\in G} d_{xy^{-1}} t F_y(t) \quad \text{if } x \neq e, \text{ and}$$
$$F_e(t) = 1 + \sum_{y\in G} d_{y^{-1}} t F_y(t).$$

In other words, they satisfy the linear system

$$\begin{cases} (d_e t - 1)F_e(t) + \sum_{y\neq e} d_{y^{-1}} t F_y(t) = -1 \\ (d_e t - 1)F_x(t) + \sum_{y\neq x} d_{xy^{-1}} t F_y(t) = 0 \quad (x \neq e) \end{cases}$$

over $\mathbb{Q}(t)$.

In order to produce an explicit formula, we enumerate the elements of G, say $G = \{x_1 = e, x_2, \dots, x_s\}$, where s is the order of G. By Kramer's rule, we obtain

$$P(T(V)_e, t) = F_e(t) = \frac{p(t)}{r(t)}, \tag{3}$$

where $p(t)$ and $r(t)$ are polynomials with integer coefficients given by

$$p(t) = \det \begin{bmatrix} -1 & d_{x_2} t & \dots & d_{x_s} t \\ 0 & d_e t - 1 & \dots & d_{x_s x_2^{-1}} t \\ \vdots & \vdots & \ddots & \vdots \\ 0 & d_{x_2 x_s^{-1}} t & \dots & d_e t - 1 \end{bmatrix}$$

and

$$r(t) = \det \begin{bmatrix} d_e t - 1 & d_{x_2} t & d_{x_3} t & \dots & d_{x_s} t \\ d_{x_2^{-1}} t & d_e t - 1 & d_{x_3 x_2^{-1}} t & \dots & d_{x_s x_2^{-1}} t \\ d_{x_3^{-1}} t & d_{x_2 x_3^{-1}} t & d_e t - 1 & \dots & d_{x_s x_3^{-1}} t \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{x_s^{-1}} t & d_{x_2 x_s^{-1}} t & d_{x_3 x_s^{-1}} t & \dots & d_e t - 1 \end{bmatrix}.$$

We end by showing that $\frac{1}{d}$ is a root of $r(t)$. Indeed, since $d = d_{x_1} + \cdots + d_{x_s}$, we have

$$r\left(\frac{1}{d}\right) = \frac{1}{d^2} \det \begin{bmatrix} d_e - d & d_{x_2} & d_{x_3} & \dots & d_{x_s} \\ d_{x_2^{-1}} & d_e - d & d_{x_3 x_2^{-1}} & \dots & d_{x_s x_2^{-1}} \\ d_{x_3^{-1}} & d_{x_2 x_3^{-1}} & d_e - d & \dots & d_{x_s x_3^{-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{x_s^{-1}} & d_{x_2 x_s^{-1}} & d_{x_3 x_s^{-1}} & \dots & d_e - d \end{bmatrix}.$$

Since for every $j = 1, \dots, s$, we have

$$\sum_{\substack{i=1 \\ i\neq j}}^{s} d_{x_i x_j^{-1}} = d - d_e,$$

the matrix above has columns adding to the zero vector. Hence its determinant is equal to zero.

□

3. Finite generation of the identity component of a linear grading

In this section we shall show that, as a corollary to Theorem 2, under some restrictive conditions, finite generation of the identity component is equivalent to the action being scalar.

Given a finite group G and a G-graded vector space, say, $V = \bigoplus_{x\in G} V_x$, we say that the grading is *trivial* if $V_x = \{0\}$ for all but one of the subspaces V_x. It is easy to see that the G-grading on V is trivial if and only if the action of $(kG)^*$ on $T(V)$ is scalar.

We start by showing that the invariants are finitely generated under a trivial grading.

Theorem 3. *Let G be a finite group and let V be a finite-dimensional space trivially graded by G. Then the identity component $T(V)_e$ of $T(V)$ under the G-grading induced by the grading on V is finitely generated.*

Proof. Suppose that $V = \bigoplus_{x\in G} V_x$ and that $x \in G$ is such that $V_x = V$, while $V_y = \{0\}$ for all $y \in G$, $y \neq x$. Then $T(V)_e = k \oplus V^{\otimes r} \oplus V^{\otimes 2r} \oplus \dots$, where r is the order of x in G. It follows that, given a basis $\{v_1, \dots, v_d\}$ of V, the subalgebra $T(V)_e$ is generated by the set of all d^r monomials of length r on the basis elements $v_1, \dots, v_d$. □

For a partial converse, we shall need the following result of Dicks and Formanek.

Lemma 4. *Let V be a finite-dimensional vector space and let H be a Hopf algebra. Suppose that V is a left H-space and consider the linear action of H on $T(V)$ induced by the action of H on V. Then the (free) subalgebra of invariants $T(V)^H$ of $T(V)$ under the action of H is a finitely generated algebra if and only if $P(T(V)^H, t)^{-1}$ is a polynomial*

Proof. The same proof of [DF82, Lemma 2.1] applies. □

Theorem 5. *Let G be a finite group and let V be a G-graded vector space, say, $V = \bigoplus_{x\in G} V_x$ with $V_e \neq \{0\}$. If the grading is not trivial, then $T(V)_e$ is not finitely generated.*

Proof. If the grading is not trivial, there exits $x \in G$, $x \neq e$, such that $V_x \neq \{0\}$. Let $W = V_e \oplus V_x$. Then W is a G-graded vector space and the canonical surjection $V \to W$ is a homomorphism of G-graded spaces. Thus, it induces a surjective homomorphism of $(kG)^*$-module algebras $T(V) \to T(W)$, which, then, restricts to a surjective algebra map $T(V)_e \to T(W)_e$. It follows that if $T(V)_e$ is finitely generated, then so is $T(W)_e$.

We have, thus, reduced the problem to considering a G-vector space $V = \bigoplus_{x\in G} V_x$ with the property that there exist $x \in G$, $x \neq e$, such that $V_y = \{0\}$ for all $y \in G \setminus \{e, x\}$, while $V_e \neq \{0\}$ and $V_x \neq \{0\}$. Applying Lemma 4 to the linear action of $(kG)^*$ on $T(V)$ induced by the G-grading on V, it suffices to show that $P(T(V)_e, t)$ is not the inverse of a polynomial. For each $y \in G$, let $d_y = \dim V_y$. We shall use formula 3 to show that $\frac{1}{d_e}$ is a root of $p(t)$ but not of $r(t)$. This implies that $(1 - d_e t)$ is a factor of $p(t)$ which does not divide $r(t)$. So $P(T(V)_e, t)$ can not be the inverse of a polynomial.

To show that $\frac{1}{d_e}$ is a root of $p(t)$, observe that $p(\frac{1}{d_e})$ is the determinant of a matrix with a row of zeros (choosing x_2 to be x^{-1} in the enumeration of the elements of

G makes the second row of this matrix null). On the other hand, $r(\frac{1}{d_e})$ equals the $\frac{d_x}{d_e}$ times the determinant of a matrix obtained from the identity matrix by a permutation of columns and, thus, is different from zero. □

References

[DNR01] S. Dăscălescu, C. Năstăsescu and Ş. Raianu, *Hopf Algebras. An Introduction*, Marcel Dekker, New York, 2001.

[DF82] W. Dicks and E. Formanek, Poincaré series and a problem of S. Montgomery, *Linear and Multilinear Algebra* **12** (1982/83), 21–30.

[FM07] V. O. Ferreira and L. S. I. Murakami, Finitely generated invariants of Hopf algebras on free associative algebras, *Linear Algebra Appl.* **420** (2007), 70–78.

[FMP04] V. O. Ferreira, L. S. I. Murakami and A. Paques, A Hopf-Galois correspondence for free algebras, *J. Algebra* **276** (2004), 407–416.

[Kh78] V. K. Kharchenko, Algebras of invariants of free algebras, *Algebra i Logika* **17** (1978), no. 4, 478–487, 491. (English translation in *Algebra and Logic* **17** (1978), no. 4, 316–321 (1979).)

[Kh84] V. K. Kharchenko, Noncommutative invariants of finite groups and Noetherian varieties, *J. Pure Appl. Algebra* **31** (1984), no. 1-3, 83–90.

[La76] D. Lane, *Free Algebras of Rank Two and Their Automorphisms*, PhD thesis, London, 1976.

[Mo93] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, Amer. Math. Soc., Providence, RI, 1993.

Department of Mathematics, Institute of Mathematics and Statistics, University of São Paulo, Caixa Postal 66281, São Paulo - SP, 05314-970, Brazil

E-mail address: `vofer@ime.usp.br, ikemoto@ime.usp.br`